The Role of Interactions between Waves and Baroclinic Critical Layers in Zombie Vortex Self-Replication

Chung-Hsiang Jiang¹, Suyang Pei¹,
Pedram Hassanzadeh²,
Aaron Wienkers¹, Caleb Levy¹
Philip S. Marcus¹

¹University of California, Berkeley
²Harvard University

66th Annual Meeting of the APS DFD
Pittsburgh, PA
25 November 2013
Motivation: Zombie Vortices in Plane Couette Flow Plus Stratification

- 3D Boussinesq equation
- Plane Couette flow with velocity $\overline{V_x} = \sigma y$
- Rotation $\Omega \hat{z}$
- Coriolis parameter $f = 2\Omega$
- Stably Stratified in vertical direction with Brunt-Väisälä frequency $N(z)$
Motivation: Zombie Vortices in Plane Couette Flow Plus Stratification

- Vortex lattice structure
- Three time scales characterized by \( f \sim N \sim |\sigma| \)
- Lack of explicit length scale
- But clearly see the uniform horizontal and vertical spacing
Goal

• Horizontal and vertical spacing of the lattice structures determined by locations where waves and critical layers have the same stream-wise wave number \((n_x)\) interact at
• Need to derive dispersion relation of linear wave with large shear
• Large, we mean \(f \sim N \sim |\sigma|\)
Poincarè Waves with Variable $N(z)$ and No Shear

- Plane wave: $e^{i(k \cdot x - \omega t)}$, WKB in $z$-direction
- Dispersion Relation
  \[ \omega^2 = \frac{N^2(z) k_\perp^2}{k^2} + \frac{f^2 k_z^2}{k^2} \]
- Local geometric form:
  \[ \tan^2 \theta \approx (\omega^2 - f^2)/(N^2(z) - \omega^2) \]
- $\theta$: the angle measured from horizontal plane
- Exact solution if $N(z) = N_0$ is constant
Poincarè Waves with Variable $N(z)$ and No Shear

- Not depends on wave vector magnitude $k$ but on wave vector direction only
- Allowed frequency ranges:
  - $f^2 \leq \omega^2 \leq N^2(z)$ internal gravity wave branch
  - $N^2(z) \leq \omega^2 \leq f^2$ inertial wave branch
- St. Andrew’s cross in two dimensional and conical wave in three dimensional for constant $N_0$
Poincarè Waves and No Shear

3D Nonlinear Simulation with a numerical wave generator $N_0/f = 1/4$ and $\sigma = 0$

2D Experiment by Mowbray and Rarity
Constant $N_0$ and $f = 0, \sigma = 0$
Poincarè Waves: No Shear with $N(z)$
Internal Gravity Wave Branch

- **Wave generator** forcing at constant $\omega = \omega_c$
- $f^2 \leq \omega^2 \leq N^2(z)$
- $N(z) = 2 \omega_c \left( 1 + \frac{z}{L_z} \right)$
- $f = 0$

Blue: theoretical ray paths predicted by WKB theory
Red: Below it, $\omega > N(z)$, the forbidden region (evanescent waves)
Poincarè Waves: No Shear with $N(z)$ Inertial Wave Branch

- **Wave generator** forcing at constant $\omega = 3 \, \omega_c$
- $N^2(z) \leq \omega^2 \leq f^2$
- $f = 2 \, \omega_c \, \sqrt{8/3}$

Blue: theoretical ray paths predicted by WKB theory
Red: Above it, $\omega < N(z)$, the forbidden region (evanescent waves)
With Background Shear
WKB in both $y$- and $z$-directions

- Dispersion Relation: $\omega = \omega_0 + k \cdot \overline{V}$
- Intrinsic (un-perturbed) frequency

$$\omega_0^2 = \frac{N^2(z) k_\perp^2}{k^2} + \frac{f^2 k_z^2}{k^2}$$

- Allowed(?) frequency ranges:
  - $f^2 \leq \omega_0^2 \leq N^2(z)$ internal gravity wave like
  - $N^2(z) \leq \omega_0^2 \leq f^2$ inertial wave like

- We always use $\overline{V} = \overline{V}_x(y) \hat{x} = \sigma y \hat{x}$
With Background Shear
WKB in both $y$- and $z$-directions

• One thing that bothers us
• Exact solution for $k_x = 0$ and constant $N_0$ is known but NOT satisfied

Exact form: $\omega_0^2 = \frac{N_0^2 k_y^2}{k_y^2 + k_z^2} + \frac{f(f - \sigma) k_z^2}{k_y^2 + k_z^2}$

• Allowed frequency ranges:
  ▪ $f(f - \sigma) \leq \omega_0^2 \leq N_0^2$
  ▪ $N_0^2 \leq \omega_0^2 \leq f(f - \sigma)$
• More importantly, one more complexity, Critical Layer, might appear
Poincarè Waves (Inertial Wave Branch)
Shear Effect in Physical Space

$\omega_z$ at $z = \frac{3}{8} l_z$
$
\vec{V} = \vec{V}_x(y) \hat{x} = \sigma y \hat{x}
$
Constant $N_0, f$ and $\sigma$

$\sigma = 0$

$\sigma = 0.01 f$

$\sigma = 0.25 f$

$\sigma = 0.75 f$
Poincarè Waves (Inertial Wave Branch) Shear Effect in Fourier Space

\[ \hat{\omega}_z \text{ at } z = \frac{3}{8} L_z \]

\[ \sigma = 0 \]

\[ \sigma = 0.01 \, f \]

\[ \sigma = 0.25 \, f \]

\[ \sigma = 0.75 \, f \]

Large shear flow prefers small stream-wise wave numbers
Critical Layer

• Rayleigh equation \( (\overline{V_x} - c)(\phi'' - k_x \phi) \) ... in unidirectional, dissipation-less shear flows with \( \overline{V} = \overline{V_x}(y) \hat{x} \)

• At location \( y_c \) where an eigenmode’s phase velocity \( c = \omega/k_x = \overline{V_x}(y_c) \), there is a critical layer

• Adding stratification, *baroclinic critical layer* occurs at \( y_c = \pm \frac{\omega}{\sigma k_x} \pm \frac{N(z)}{\sigma k_x} \)
Wave Packets and Critical Layers

Constant $N(z) = N_0$

- $n_x = 1$ C.L.
- $n_x = 2$ waves
- $n_x = 2$ C.L.

Linear $N(z) = N_0 \left( 1 + 2z \right)$

- $n_x = 1$ C.L.
- $n_x = 2$ C.L.
- $n_x = 3$ C.L.
- $n_x = 1$ Waves

$\omega_z$ at $x = 0$ plane. Dashed lines indicate Baroclinic Critical Layers. $\sigma/f = 3/4$ and $f/N_0 = 2/3$
New Small $k_x$ Dispersion Relation
WKB Plus Small $k_x$ Approximations

• $k_x \ll k_y \sim k_z$ for large shear $\sigma$
• Dispersion Relation:
  \[ \omega = \omega_0 + k \cdot \bar{V} \]
• Intrinsic frequency:
  \[ \omega_0^2 = \frac{N^2(z) k_y^2}{k_y^2 + k_z^2} + \frac{f(f - \sigma) k_z^2}{k_y^2 + k_z^2} \]
  \[ \approx N^2(z) \sin^2 \theta + f(f - \sigma) \cos^2 \theta \]
• Allowed frequency ranges:
  • $f(f - \sigma) \leq \omega_0^2 \leq N^2(z)$
  • $N^2(z) \leq \omega_0^2 \leq f(f - \sigma)$. 
Cherry Picking Validation:
Local Geometric Forms

- Small $k_x$ Dispersion Relation:
  \[
  \omega_0^2(y_l) \approx N^2(z_l) \sin^2 \theta + f(f - \sigma) \cos^2 \theta
  \]
  \[
  \left( \frac{dz}{dy} \right)^2 \approx \tan^2 \theta \approx \frac{\omega_0^2(y_l) - f(f - \sigma)}{N^2(z_l) - \omega_0^2(y_l)}
  \]

- Textbook dispersion relation:
  \[
  \omega_0^2(y_l) \approx N^2(z_l) \sin^2 \theta + f^2 \cos^2 \theta
  \]
  \[
  \left( \frac{dz}{dy} \right)^2 \approx \tan^2 \theta \approx \frac{\omega_0^2(y_l) - f^2}{N^2(z_l) - \omega_0^2(y_l)}
  \]
Validation of Small $k_x$ Dispersion Relation
Constant $N_0$ Case

Small $k_x$ Dispersion Relation

$n_x = 1$ waves

$n_x - 2$ waves

Textbook Dispersion Relation

$n_x = 1$ waves

$n_x = 2$ waves

$v_z$ at $x = 0$ plane

Blue: projection of rays onto $x = 0$ plane
Validation of Small $k_x$ Dispersion Relation
Linear $N(z)$ Case

Small $k_x$ Dispersion Relation

Textbook Dispersion Relation

$v_z$ at $x = 0$ plane

Blue: projection of rays onto $x = 0$ plane
Conclusion

• The new small $k_x$ dispersion relation works much better compared to traditional one
• Both horizontal and vertical spacing of the lattice structures determined by locations where waves and critical layers have the same stream-wise wave number ($n_x$) interact at
• More serious validation of small $k_x$ dispersion relation is ongoing